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A VARIATIONAL PRINCIPLE FOR THE DYNAMIC ANALYSIS OF CONTINUA BY HYBRID FINITE ELEMENT METHOD

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Abstract—The hybrid finite element procedure proposed by Pian has been extended for dynamic analysis of continua via a variational principle.

The formulation is then illustrated by frequency analysis of a plate in flexural motion.

1. INTRODUCTION

IN THE dynamic analysis of continua by finite element approach the system stiffness and mass matrices are often obtained by the application of Hamilton's principle. In this approach the kinetic and strain energies of an element are expressed in terms of some assumed displacement functions and it is well known that for monotonic convergence to the actual solution it is necessary that these functions should satisfy the compatibility conditions at the boundaries of the elements. Now the construction of displacement functions for the interior of an element which will also ensure compatibility at the boundaries is, in most cases, rather complicated even for elements with simple geometry.

In 1964 Pian [1] proposed an approach, for static analysis of continua, by means of which compatible stiffness matrices can be obtained without undue difficulties. In Pian's approach [2, 3] one assumes stresses within the element and displacements on its boundary —hence the word hybrid.

In 1967 Dungar *et al.* [4] used the hybrid method for vibration analysis of plates and shells. In this paper an interesting procedure was described for the derivation of the mass matrix of an element. The displacements within the element were obtained by interpolations from the boundary displacements. Depending on the boundaries used, a different mass matrix was obtained and the mean of these was taken as the representative mass matrix. While this method yielded results which were in good agreement with experimental results it must be conceded that the procedure lacks mathematical justification.

Recently Pian [5] showed that his method for deriving stiffness matrices of elements is actually an application of a variational principle. In this paper we generalize Pian's variational principle for dynamic analysis of continua and show that consistent inertia properties of the elements can be obtained from the assumed stress functions within the element.

2. EQUATIONS OF ELASTO-MECHANICS

Consider a body V subjected to some prescribed dynamic forces F_i. If σ_{ij} are stress components satisfying the equations of dynamic equilibrium then we can write:

$$\sigma_{ij,j}+F_i = \dot{r}_i \qquad \text{in } V \text{ at all } t \qquad i = 1, 2, 3 \tag{1}$$

where

$$,j = \frac{\partial}{\partial x_j} \qquad \cdot = \frac{\partial}{\partial t}$$

and r_i denotes the inertia impulse vector. Let us write equations (1) in the following form:

$$\frac{\partial}{\partial t}\left[\int_0^t (\sigma_{ij,j}+F_i-\dot{r}_i)\,\mathrm{d}t+(\sigma_{ij,j}+F_i-\dot{r}_i)_0\right] = 0 \tag{2}$$

where the bracket with subscript 0 denotes initial state of the body and clearly it is equal to zero. From equation (2) it is apparent that the equation of dynamic equilibrium may be written as:

(D) $\qquad \tau_{ij,j}+f_i = r_i \qquad i = 1, 2, 3 \qquad$ in V at all t

where

$$\dot{\tau}_{ij} = \sigma_{ij}$$

and

$$\dot{f}_i = F_i.$$

We will refer to τ_{ij} as the impulse tensor field. A body will generally be subjected to both kinematical and dynamical boundary conditions. Thus if on one part of the surface, S_u, the velocities u_i are prescribed and on another part, S_τ, the surface impulses are prescribed then these conditions can be expressed as:

(K.B.C.) $\qquad u_i = \bar{u}_i \quad$ on $S_u \qquad i = 1, 2, 3$

(D.B.C.) $\qquad \tau_{ij}n_j = \bar{t}_i \quad$ on $S_\tau \qquad i = 1, 2, 3.$

Clearly t_i is the surface traction vector and n_j is the unit external normal vector and the barred quantities are prescribed.

Denoting the strain tensor by ε_{ij} then the compatibility equations may be written as:

(C) $\qquad \dot{\varepsilon}_{ij} = \frac{1}{2}(u_{i,j}+u_{j,i}).$

We now define the kinetic energy function as:

$$T = \int_v T_1(r_i)\,\mathrm{d}V \quad \text{such that} \quad \frac{\partial T_1}{\partial r_i} = u_i \tag{3}$$

or

$$T_1 = \int u_i(r_i)\,\mathrm{d}r_i.$$

For Newtonian mechanics $r_i = \rho u_i$ and hence

$$T_1 = \frac{1}{2\rho}\delta_{ij}r_i r_j, \qquad i,j = 1, 2, 3$$

where ρ denotes the density of the material, and δ_{ij} is the kronecker delta. Similarly we define the complementary strain energy as:

$$S^* = \int_v S_1^*(\dot{\tau}_{ij})\,dV \quad \text{such that} \quad \frac{\partial S_1^*}{\partial \dot{\tau}_{ij}} = \varepsilon_{ij}.$$

For linear analysis S_1^* is given by

$$S_1^* = \tfrac{1}{2}C_{ijkl}\dot{\tau}_{ij}\dot{\tau}_{kl}$$

where C_{ijkl} is the compliance tensor which is positive definite and $C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}$.

3. THE VARIATIONAL PRINCIPLE

We derive Pian's variational principle by suitably modifying the complementary variational principle for elastodynamics.

The variational principle which is complementary to Hamilton's principle can be stated as (see Ref. [8–10])

$$\pi_c = \int_{t_1}^{t_2} \left[T(r_i) - S^*(\sigma_{ij}) - \int_{S_u} t_i \bar{u}_i \,ds \right] dt. \tag{4}$$

The integral over S_u represents the work of the specified surface velocities $\bar{u}_i$.

As is well known the competing stress fields to extremise π_c are required to:

(i) satisfy the equations of dynamic equilibrium (D) in V at all t,

(ii) be in conformity with the dynamic boundary conditions on S_τ at all t.

Now in the finite element procedure the body V is fictitiously cut into p smaller subregions V_p. The application of the complementary variational principle, for the finite element analysis, would then require that:

(i) Equilibrium be satisfied in each V_p.

(ii) Equilibrium be satisfied on the inter element surfaces and on surfaces S_{τ_p} (for the appropriate elements).

While the satisfaction of the first requirement on its own is straightforward the simultaneous satisfaction of both requirements poses the major difficulty in the application of the complementary variational principle to finite element formulation. Pian's formulation removes this difficulty by posing the second requirement as equations of constraint and incorporating it into the functional π_c by means of Lagrange multipliers.

The equilibrium of two elements I and II along a mutual surface $S_N(\mathrm{I, II})$ requires that

$$(t_i)_{\mathrm{I}} + (t_i)_{\mathrm{II}} = 0 \quad \text{on} \quad S_N(\mathrm{I, II}) \text{ at all } t.$$

On multiplying this equation of constraint by λ_i we can write

$$\int_{S_N(\mathrm{I,II})} \lambda_i[(t_i)_{\mathrm{I}} + (t_i)_{\mathrm{II}}]\,ds = 0$$

or

$$\int_{S_N(\mathrm{I,II})} \lambda_i(t_i)_{\mathrm{I}}\,ds + \int_{S_N(\mathrm{I,II})} \lambda_i(t_i)_{\mathrm{II}}\,ds = 0.$$

When all the interelement surfaces have been considered the constraint equations may be incorporated into π_c and then the discretised form of the complementary energy functional can be written as

$$\pi = \int_{t_1}^{t_2} \sum_p \left[T_p - S_p^* - \int_{S_{up}} t_i \bar{u}_i \, ds - \int_{S_{Np}} u_i t_i \, ds \right] dt \tag{5}$$

where S_{up} denotes that part of the surface of V_p on which the velocities are prescribed and S_{Np} denotes the interelement surface of V_p.

It can easily be shown that on equating the variation of π to zero one is led to:

(i) compatibility of velocities in each V_p,

(ii) compatibility of velocities on the interelement surfaces and on surfaces S_{up}, for the appropriate elements.

Further one finds that the lambda multipliers are equal to the (compatible) velocities on the interelement surfaces. Thus the functional in equation (5) can now be written as

$$\pi = \int_{t_1}^{t_2} \sum_p \left[T_p - S_p^* - \int_{S_{up}} t_i \bar{u}_i \, ds - \int_{S_{Np}} u_i t_i \, ds \right] dt. \tag{6}$$

Now we have considered the surface of an element in three mutually exclusive parts, i.e.

$$S_p = S_{up} + S_{\tau p} + S_{Np}. \tag{7}$$

As stated earlier on S_{up} velocities are prescribed, on $S_{\tau p}$ impulses are prescribed while on S_{Np} neither displacements nor impulses are prescribed.

Using equation (7) we follow Pian and express S_{Np} in terms of the other surfaces. Then equation (6) can be written as

$$\pi = \int_{t_1}^{t_2} \sum \left[T_p - S_p^* - \int_{S_p} t_i u_i \, ds + \int_{S_{\tau p}} \bar{t}_i u_i \, ds \right] dt. \tag{8}$$

In the above functional both impulses τ_{ij} and the velocities u_i are subject to variations.

We consider a class of admissible impulses each member of which:

(i) possesses continuous first derivatives in each V_p,

(ii) satisfies equation of dynamic equilibrium (D) in V_p,

(iii) is prescribed at arbitrary times t_1 and t_2.

Further we consider a class of admissible surface velocities each member of which:

(i) is continuous on S_p,

(ii) satisfies the kinematic boundary condition (K.B.C.),

(iii) satisfies compatibility along the mutual surfaces S_{Np},

(iv) is prescribed at arbitrary times t_1 and t_2.

The solution in addition to the above conditions satisfies:

(i) compatibility equations (C) in each V_p,

(ii) continuity of displacements across the elemental boundaries,

(iii) equilibrium of elements V_p with each other along mutually common boundaries S_{Np},

(iv) dynamic boundary conditions (D.B.C.).

Theorem

Amongst all the admissible impulses and velocities the true solution is distinguished by the stationary conditions of π. To prove the theorem we equate the variation of π to zero and thence deduce the conditions associated with the true solution. Now

$$\delta\pi = \int_{t_1}^{t_2} \sum_p \left[\delta T_p - \delta S_p^* - \int_{S_p} u_i n_j \delta\tau_{ij}\, ds + \int_{S_{\tau p}} (-\tau_{ij} n_j + \bar{t}_i)\delta u_i\, ds - \int_{S_{Np}} \tau_{ij} n_j \delta u_i\, ds \right] dt = 0. \tag{9}$$

Now from foregoing it will be apparent that the variation of velocities over $S_{\tau p}$ is independent from that over S_{Np}.† However variations of velocities over S_{Np} for adjoining elements are not independent and hence for $\delta\pi = 0$ we deduce

$$\tau_{ij} n_j = \bar{t}_i \quad \text{on} \quad S_{\tau p} \qquad \text{(D.B.C.)}$$

$$\sum_p \tau_{ij} n_j = 0 \quad \text{on} \quad S_{Np} \qquad \text{(Interelement equilibrium).}$$

Consider now the term

$$\int_{t_1}^{t_2} \sum_p \left[\delta T_p - \delta S_p^* - \int_{S_p} u_i n_j \delta\tau_{ij}\, ds \right] dt = 0. \tag{10}$$

For an element V_p bounded by surface S_p we have

$$\delta T_p = \int_{V_p} \frac{\partial T_1}{\partial r_i} \delta r_i\, dV$$

but $\partial T_1/\partial r_i = u_i$ by definition and $\delta r_i = \delta\tau_{ij,j}$ by (D) since the body impulse vector f_i is prescribed. Thus we can write

$$\delta T_p = \int_{V_p} (u_i \delta\tau_{ij})_{,j}\, dV - \int_{V_p} u_{i,j} \delta\tau_{ij}\, dV \tag{11}$$

applying the divergence theorem to the first term results in

$$\delta T_p = \int_{S_p} u_i \delta\tau_{ij} n_j\, ds - \int_{V_p} u_{i,j} \delta\tau_{ij}\, dV. \tag{12}$$

Likewise

$$\delta S_p^* = \int_{V_p} \frac{\partial S_p^*}{\partial \dot{\tau}_{ij}} \delta\dot{\tau}_{ij}\, dV$$

but $\partial S_p^*/\partial\dot{\tau}_{ij} = \varepsilon_{ij}$ by definition therefore

$$\delta S_p^* = \int_{V_p} \varepsilon_{ij} \delta\dot{\tau}_{ij}\, dV. \tag{13}$$

Now equations (12) and (13) can be substituted into equation (10). At this stage it is of importance to realise that u_i under the integral S_p in equation (12) has been obtained from kinetic energy which is itself derived from a set of assumed admissible impulses inside the element. On the other hand u_i under the integral S_p in equation (10) is an admissible boundary velocity which is assumed independently from the admissible impulses. The two u_i's will be, in general different, i.e. the continuity of displacements across a boundary surface is not satisfied *explicitly* but we note that extremisation of π requires that this condition be satisfied *implicitly*. After the stated substitution we will be left with the term

$$\int_{t_1}^{t_2} \sum_p \left[\int_{V_p} -u_{i,j} \delta\tau_{ij}\, dV - \int_{V_p} \varepsilon_{ij} \delta\dot{\tau}_{ij}\, dV \right] dt = 0. \tag{14}$$

† Except along the line separating these surfaces.

Now due to the symmetry of τ_{ij} we can write

$$u_{i,j}\delta\tau_{ij} = \tfrac{1}{2}(u_{i,j}+u_{j,i})\delta\tau_{ij}. \tag{15}$$

Then integrating the second integral by parts over time yields the result

$$\int_{t_1}^{t_2}\sum_p\left[\int_{V_p}(-\tfrac{1}{2}(u_{i,j}+u_{j,i})+\dot{\varepsilon}_{ij})\delta\tau_{ij}\,\mathrm{d}V\right]\mathrm{d}t-\sum_p\int_{V_p}\varepsilon_{ij}\delta\tau_{ij}\big|_{t_1}^{t_2}\,\mathrm{d}V = 0. \tag{16}$$

The last term vanishes since $\delta\tau_{ij} = 0$ at t_1 and t_2. The vanishing of first term requires

$$\dot{\varepsilon}_{ij} = \tfrac{1}{2}(u_{i,j}+u_{j,i}),\ \text{compatibility in } V_p. \tag{17}$$

4. FINITE ELEMENT FORMULATION

In this section we develop the relevant equations of analysis in matrix notation. For sake of clarity in presentation we consider the case when the body forces are absent. The procedure for inclusion of body forces is straight forward and has been outlined by Pian [5].

We now consider the less general case of harmonic vibrations. Thus we let the prescribed surface impulses be harmonic in nature with frequency ω i.e.

$$\bar{t}_i(x, y, z, t) = \hat{\bar{t}}_i(x, y, z)\sin\omega t \tag{18}$$

where $\hat{\bar{t}}_i$ indicates the amplitude of $\bar{t}_i$. Under steady state conditions the velocity vector u_i and impulse tensor τ_{ij} will also vary harmonically with frequency ω and when these quantities are substituted into equation (8) we obtain

$$\pi = \int_{t_1}^{t_2}\sin^2\omega t\,\mathrm{d}t\sum_p\left[\hat{T}_p-\int_{S_p}\hat{t}_i\hat{u}_i\,\mathrm{d}s+\int_{S_{\tau p}}\hat{\bar{t}}_i\hat{u}_i\,\mathrm{d}s\right]-\int_{t_1}^{t_2}\cos^2\omega t\,\mathrm{d}t\sum_p\omega^2\hat{S}_p^*. \tag{19}$$

The capped quantities indicating the amplitudes. Now if the instances t_1 and t_2 are so chosen that $t_2 = t_1+2\pi/\omega$ then

$$\int_{t_1}^{t_1+2\pi/\omega}\sin^2\omega t\,\mathrm{d}t = \int_{t_1}^{t_1+2\pi/\omega}\cos^2\omega t\,\mathrm{d}t = \mathrm{C} \tag{20}$$

where C is a constant. Then the functional π takes the following form

$$\pi = C\hat{\pi}$$

where

$$\hat{\pi} = \sum_p\left[\hat{T}_p-\omega^2S_p^*-\int_{S_p}\hat{t}_i\hat{u}_i\,\mathrm{d}s+\int_{S_{\tau p}}\hat{\bar{t}}_i\hat{u}_i\,\mathrm{d}s\right]. \tag{21}$$

Now let the admissible impulses $\{\tau\}$ be expressed in terms of m undetermined coefficients $\{\beta\}$. The β's should be so chosen as to satisfy the dynamic equilibrium equation (D). The solution of (D) consists of a homogeneous and a particular part, thus we write:

$$\{\tau\} = [A]_1\{\beta\}_1+[A]_2\{\beta\}_2. \tag{22}$$

The first term $[A]_1\{\beta\}_1$ is the homogeneous solution while $[A]_2\{\beta\}_2$ represents the particular solution. The surface impulses $t_i(=\tau_{ij}n_j)$ can be expressed in terms of β's as

$$\{t\} = [B]_1\{\beta\}_1+[B]_2\{\beta\}_2. \tag{23}$$

Also the inertia impulse vector $r_i(=\tau_{ij,j})$ can be expressed in terms of β's as

$$\{r\} = [G]\{\beta\}_2 \tag{24}$$

where $[G]$ is obtained from $[A]_2$ by the appropriate differentiation operations. Finally we express the boundary velocities u_i in terms of some nodal velocities $\{\dot{q}\}$ i.e.

$$\{u\} = [L]\{\dot{q}\}. \tag{25}$$

Then the functional $\hat{\pi}$, for a linear system, can be expressed as:

$$\hat{\pi} = \sum_p [\beta_1^t \beta_2^t]\left(\frac{1}{2}\begin{bmatrix} [0] & [0] \\ [0] & [N]_{22} \end{bmatrix}\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} - \frac{\omega^2}{2}\begin{bmatrix} [\phi]_{11} & [\phi]_{12} \\ [\phi]_{21} & [\phi]_{22} \end{bmatrix}\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} - \begin{bmatrix} [R]_1 \\ [R]_2 \end{bmatrix}\{\dot{q}\}\right) + \{S\}^t\{\dot{q}\} \tag{26}$$

where

$$\begin{aligned}
[N]_{22} &= \int_{V_p} \frac{1}{\rho}[G]^t[G]\,\mathrm{d}V & [\phi]_{11} &= \int_{V_p} [A]_1^t[C][A]_1\,\mathrm{d}V \\
[\phi]_{12} = [\phi]_{21}^t &= \int_{V_p} [A]_1^t[C][A]_2\,\mathrm{d}V & [\phi]_{22} &= \int_{V_p} [A]_2^t[C][A]_2\,\mathrm{d}V \\
[R]_1 &= \int_{S_p} [B]_1^t[L]\,\mathrm{d}s & [R]_2 &= \int_{S_p} [B]_2^t[L]\,\mathrm{d}s \\
\{S\}^t &= \int_{S_{\tau p}} \{\hat{t}\}[L]\,\mathrm{d}s
\end{aligned} \tag{27}$$

and matrix $[C]$ is the compliance tensor. On taking the variation of $\hat{\pi}$ w.r.t. β's and the unspecified $\dot{q}$'s we obtain

$$\delta\hat{\pi} = \sum_p \delta[\beta_1^t \beta_2^t]\left\{\left(\begin{bmatrix} [0] & [0] \\ [0] & [N]_{22} \end{bmatrix} - \omega^2\begin{bmatrix} [\phi]_{11} & [\phi]_{12} \\ [\phi]_{21} & [\phi]_{22} \end{bmatrix}\right)\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} - \begin{bmatrix} [R]_1 \\ [R]_2 \end{bmatrix}\{\dot{q}\}\right\} + \sum_p\left(-[\beta_1^t \quad \beta_2^t]\begin{bmatrix} [R]_1 \\ [R]_2 \end{bmatrix} + \{S\}^t\right)\delta\{\dot{q}\}. \tag{28}$$

Now since the variations of β's and $\dot{q}$'s are arbitrary and independent of each other, each sum in the above expression must vanish for $\delta\hat{\pi} = 0$. Further since the β's of one element are independent from those of another element, while the $\dot{q}$'s are not, we deduce that for $\delta\hat{\pi} = 0$.

$$\sum_p \left(-[\beta_1^t \beta_2^t]\begin{bmatrix} [R]_1 \\ [R]_2 \end{bmatrix} + \{S\}^t\right)\delta\{\dot{q}\} = 0 \tag{29}$$

and

$$\left\{\begin{bmatrix} [0] & [0] \\ [0] & [N]_{22} \end{bmatrix} - \omega^2\begin{bmatrix} [\phi]_{11} & [\phi]_{12} \\ [\phi]_{21} & [\phi]_{22} \end{bmatrix}\right\}\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} [R]_1 \\ [R]_2 \end{bmatrix}\{\dot{q}\}. \tag{30}$$

From the latter equation we obtain

$$\{\beta\} = [D]^{-1}[R]\{\dot{q}\} \tag{31}$$

where

$$\{\beta\} = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \quad [D] = \left(\begin{bmatrix} [0] & [0] \\ [0] & [N]_{22} \end{bmatrix} - \omega^2 \begin{bmatrix} [\phi]_{11} & [\phi]_{12} \\ [\phi]_{21} & [\phi]_{22} \end{bmatrix} \right)$$

and

$$[R] = \begin{bmatrix} [R]_1 \\ [R]_2 \end{bmatrix}.$$

On substituting the above expression for $\{\beta\}$ into $\hat{\pi}$ we find

$$\hat{\pi} = \sum_p [-\tfrac{1}{2}\{\dot{q}\}^t[R]^t[D]^{-1}[R]\{\dot{q}\} + \{S\}^t\{\dot{q}\}]$$

or

$$\hat{\pi} = \sum_p [-\tfrac{1}{2}\{\dot{q}\}^t[M]\{\dot{q}\} + \{S\}^t\{\dot{q}\}] \tag{32}$$

where

$$[M] = [R]^t[D]^{-1}[R]. \tag{33}$$

In equation (32) we note that the first part describes the complementary kinetic energy of the system while the second part yields the work done by the prescribed nodal impulses. Clearly $[M]$ is the mass matrix of the element† and since it is frequency dependent we will refer to it, for lack of a better name, as the dynamic mass matrix.

Now the nodal velocities $\{\dot{q}\}$ for different elements are not independent. Hence a transformation is required to relate the element nodal velocities to a column of independent global velocities,

$$[\{\dot{q}\}_1^t\{\dot{q}\}_2^t \ldots \{\dot{q}\}_p^t] = \{\dot{q}\}^t[J]^t \tag{34}$$

where $[J]$ is the, commonly called, connection matrix.

Finally, $\hat{\pi}$ can be written as:

$$\hat{\pi} = -\tfrac{1}{2}\{\dot{\mathbf{q}}\}^t[\mathbf{M}]\{\dot{\mathbf{q}}\} + \{\mathbf{S}\}^t\{\dot{\mathbf{q}}\} \tag{35}$$

where

$$[\mathbf{M}] = [J]^t([M]_1[M]_2 \ldots [M]_p)[J] \tag{36}$$

is the dynamic mass matrix of the assembled system and

$$\{\mathbf{S}\}^t = [\{S\}_1^t\{S\}_2^t \ldots \{S\}_p^t][J] \tag{37}$$

is the applied generalized nodal impulses consistent with the assumed velocity functions.

Variation of $\hat{\pi}$ w.r.t. $\dot{\mathbf{q}}$'s will now yield

$$[\mathbf{M}]\{\dot{\mathbf{q}}\} = \{\mathbf{S}\} \tag{38}$$

which is the velocity momentum relation for the system. For free vibrations $\{\mathbf{S}\} = \{0\}$ and so the natural frequencies of the system will be given by

$$|\mathbf{M}| = 0.$$

†Matrix $[M]$ is of course *not* the same as the mass matrix obtained by Hamilton's principle. The relation between the two is analogous to that between the inverse of dynamic stiffness matrix and the flexibility matrix. For a detailed discussion see Ref. 6.

Clearly the degeneracy or otherwise of the dynamic mass matrix of an element should depend upon the frequency only. Thus in equation (33) it is necessary that the $[R]$ matrix possess more rows than columns. This is equivalent to requiring that in general the number of β's of an element should be in excess of the number of $\dot{q}$'s of the same element. There will be some relations amongst the rows of $[R]$ due to the self equilibrating stresses but the columns of $[R]$ should of course be independent.

5. ILLUSTRATIVE EXAMPLE

The procedure described above has been used to calculate the first few natural frequencies of a plate in flexural vibration, using rectangular elements.

The consideration of balance of forces on an infinitesimal element of the plate leads to the following equations of equilibrium

$$m_{x,x} + m_{xy,y} = s_x \tag{39}$$

$$m_{xy,x} + m_{y,y} = s_y \tag{40}$$

$$s_{x,x} + s_{y,y} = r \tag{41}$$

where

$\dot{m}_x$ = bending moment about x axis
$\dot{m}_y$ = bending moment about y axis
$\dot{m}_{xy}$ = twisting moment about the normal to the edge of the element
$\dot{s}_x$ = shear force on an edge normal to x axis
$\dot{s}_y$ = shear force on an edge normal to the y axis
r = linear impulse normal to the plane of the plate.

In the present analysis $r = \rho \dot{w}$ where w is the transverse velocity of the plate. The state of impulse within a rectangular element of dimensions a and b can be approximated by

$$\begin{bmatrix} m_x \\ m_y \\ m_{xy} \\ s_x \\ s_y \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \bar{x} & 0 & 0 & \bar{y} & 0 & 0 & 0 & 0 & \bar{y}^2 & 0 & \bar{x}\bar{y} & 0 \\ 0 & 1 & 0 & 0 & \bar{x} & 0 & 0 & \bar{y} & 0 & \bar{x}^2 & 0 & 0 & 0 & 0 & \bar{x}\bar{y} \\ 0 & 0 & 1 & 0 & 0 & \bar{x} & 0 & 0 & \bar{y} & 0 & \bar{x}^2 & 0 & \bar{y}^2 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{a} & 0 & 0 & 0 & 0 & \frac{1}{b} & 0 & 0 & 0 & \frac{2\bar{y}}{b} & \frac{\bar{y}}{a} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{a} & 0 & \frac{1}{b} & 0 & 0 & \frac{2\bar{x}}{a} & 0 & 0 & 0 & \frac{\bar{x}}{b} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \cdot \\ \cdot \\ \cdot \\ \beta_{15} \end{bmatrix}$$

$$+ \begin{bmatrix} \bar{x}^2 & 0 & 0 & \bar{x}^3 & 0 & \bar{x}^2\bar{y} & 0 & 0 & 0 \\ 0 & \bar{y}^2 & 0 & 0 & \bar{y}^3 & 0 & 0 & \bar{x}\bar{y}^2 & 0 \\ 0 & 0 & \bar{x}\bar{y} & 0 & 0 & 0 & \bar{x}^2\bar{y} & 0 & \bar{x}\bar{y}^2 \\ \frac{2\bar{x}}{a} & 0 & \frac{\bar{x}}{b} & \frac{3\bar{x}^2}{a} & 0 & \frac{2\bar{x}\bar{y}}{a} & \frac{\bar{x}^2}{b} & 0 & \frac{2\bar{x}\bar{y}}{b} \\ 0 & \frac{2\bar{y}}{b} & \frac{\bar{y}}{a} & 0 & \frac{3\bar{y}^2}{b} & 0 & \frac{2\bar{x}\bar{y}}{a} & \frac{2\bar{x}\bar{y}}{b} & \frac{\bar{y}^2}{a} \end{bmatrix} \begin{bmatrix} \beta_{16} \\ \cdot \\ \cdot \\ \cdot \\ \beta_{24} \end{bmatrix} \tag{42}$$

where $\bar{x}$ and $\bar{y}$ are respectively x/a and y/b.

It will be noted that the first fifteen β's are involved in the homogeneous solution of equations of equilibrium (i.e. $r = 0$) while the remainder nine β's are concerned with the particular solution.

Using equation (42) one can evaluate the kinetic energy and the complementary strain energy of the element from the expressions

$$T = \frac{1}{2\rho h}\int_0^a \int_0^b r^2(x, y)\,\mathrm{d}x\,\mathrm{d}y \tag{43}$$

$$S^* = \frac{1}{2D(1-v^2)}\int_0^a \int_0^b [\dot{m}_x^2 + \dot{m}_y^2 - 2v\dot{m}_x\dot{m}_y + 2(1+v)\dot{m}_{xy}]\,\mathrm{d}x\,\mathrm{d}y \tag{44}$$

where h is the thickness of the element, v is Poisson's ratio and $D = (12h^3/E(1-v^2))$ is the flexural rigidity of the element.

The corresponding matrices $[N]_{22}$ and $[\phi]_{11}$, $[\phi]_{12}$ and $[\phi]_{22}$ are given in Tables 1–4.

TABLE 1

$$[N]_{22} = \frac{1}{\rho h \alpha b^2}\begin{bmatrix} 4\alpha^4 & & & & & & & & \\ 4\alpha^2 & 4 & & & & & \text{Symmetric} \quad \alpha = \frac{b}{a} & & \\ 4\alpha^3 & 4\alpha & 4\alpha^2 & & & & & & \\ 6\alpha^4 & 6\alpha^2 & 6\alpha^3 & 12\alpha^4 & & & & & \\ 6\alpha^2 & 6 & 6\alpha & 9\alpha^2 & 12 & & & & \\ 2\alpha^4 & 2\alpha^2 & 2\alpha^3 & 3\alpha^4 & 4\alpha^2 & \frac{4}{3}\alpha^4 & & & \\ 4\alpha^3 & 4\alpha & 4\alpha^2 & 8\alpha^3 & 6\alpha & 2\alpha^3 & \frac{16}{3}\alpha^2 & & \\ 2\alpha^2 & 2 & 2\alpha & 4\alpha^2 & 3 & \alpha^2 & \frac{8}{3}\alpha & \frac{4}{3} & \\ 4\alpha^3 & 4\alpha & 4\alpha^2 & 6\alpha^3 & 8\alpha & \frac{8}{3} & 4\alpha^2 & 2\alpha & \frac{16}{3}\alpha^2 \end{bmatrix}$$

In the flexure of plate the interelement compatibility calls for matching of transverse velocity w of the plate and the two independent slopes of w. For a rectangular element as shown in Fig. 1, these conditions can be satisfied by assuming, for instance for edge AB,

$$w = c_1 + c_2\bar{x} + c_3\bar{x}^2 + c_4\bar{x}^3 \tag{45}$$

$$w_{,y} = d_1 + d_2\bar{x}. \tag{46}$$

The four constants in the expression for w can be evaluated in terms of two linear nodal velocities $\dot{q}_1$ and $\dot{q}_4$ and two angular nodal velocities $\dot{q}_2$ and $\dot{q}_5$. Likewise the constants d_1 and d_2 can be expressed in terms of nodal angular velocities $\dot{q}_3$ and $\dot{q}_6$. In this way velocities along all four edges can be expressed in terms of 12 nodal velocities. The matrix $[L]$ is given in Table 5. Next from the assumed internal state of impulse, the boundary impulses can be obtained and expressed via matrices $[B]_1$ and $[B]_2$ [see equation (23)]. Finally the matrices $[R]_1$ and $[R]_2$ can be obtained from the expressions given in equation (27). These matrices are given in Tables 6 and 7.

TABLE 2

$$[\phi]_{11} = \frac{1}{D(1-\nu^2)} \left[\begin{array}{ccccccccccccccc}
1 & & & & & & & & & & & & & & \\
-\nu & 1 & & & & & & & & & & & & & \\
0 & 0 & \bar{\nu} & & & & & & & & \multicolumn{5}{l}{\text{Symmetric}} \\
\frac{1}{2} & \frac{-\nu}{2} & 0 & \frac{1}{3} & & & & & & & \multicolumn{5}{l}{(\bar{\nu} = 1+\nu)} \\
\frac{-\nu}{2} & \frac{1}{2} & 0 & \frac{-\nu}{3} & \frac{1}{3} & & & & & & & & & & \\
0 & 0 & \frac{\bar{\nu}}{2} & 0 & 0 & \frac{\bar{\nu}}{3} & & & & & & & & & \\
\frac{1}{2} & \frac{-\nu}{2} & 0 & \frac{1}{4} & \frac{-\nu}{4} & 0 & \frac{1}{3} & & & & & & & & \\
\frac{-\nu}{2} & \frac{1}{2} & 0 & \frac{-\nu}{4} & \frac{1}{4} & 0 & \frac{-\nu}{3} & \frac{1}{3} & & & & & & & \\
0 & 0 & \frac{\bar{\nu}}{2} & 0 & 0 & \frac{\bar{\nu}}{4} & 0 & 0 & \frac{\bar{\nu}}{3} & & & & & & \\
\frac{-\nu}{3} & \frac{1}{3} & 0 & \frac{-\nu}{4} & \frac{1}{4} & 0 & \frac{-\nu}{6} & \frac{1}{6} & 0 & \frac{1}{5} & & & & & \\
0 & 0 & \frac{\bar{\nu}}{3} & 0 & 0 & \frac{\bar{\nu}}{4} & 0 & 0 & \frac{\bar{\nu}}{6} & 0 & \frac{\bar{\nu}}{5} & & & & \\
\frac{1}{3} & \frac{-\nu}{3} & 0 & \frac{1}{6} & \frac{-\nu}{6} & 0 & \frac{1}{4} & \frac{-\nu}{4} & 0 & \frac{-\nu}{9} & 0 & \frac{1}{5} & & & \\
0 & 0 & \frac{\bar{\nu}}{3} & 0 & 0 & \frac{\bar{\nu}}{6} & 0 & 0 & \frac{\bar{\nu}}{4} & 0 & \frac{\bar{\nu}}{9} & 0 & \frac{\bar{\nu}}{5} & & \\
\frac{1}{4} & \frac{-\nu}{4} & 0 & \frac{1}{6} & \frac{-\nu}{6} & 0 & \frac{1}{6} & \frac{-\nu}{6} & 0 & \frac{-\nu}{8} & 0 & \frac{1}{8} & 0 & \frac{1}{9} & \\
\frac{-\nu}{4} & 0 & 0 & \frac{-\nu}{6} & \frac{1}{6} & 0 & \frac{-\nu}{6} & \frac{1}{6} & 0 & \frac{1}{8} & 0 & \frac{-\nu}{8} & 0 & \frac{-\nu}{9} & \frac{1}{9}
\end{array} \right]$$

The above evaluated matrices were then employed to derive the matrix $[D]$ of the element and finally the dynamic mass matrix $[M]$ of the element was calculated for various frequencies ω. In Table 8 the calculated frequencies of a clamped rectangular plate have been given. Denoting the side lengths of the rectangular plate by l_x and l_y, the natural frequencies can be evaluated for various side ratios. Those in Table 8 are for $l_x/l_y = \frac{1}{2}$. For purposes of comparison natural frequencies given by Warburton [11], using beam eigenfunctions in Rayleigh–Ritz analysis, have also been given. These latter frequencies are well within 1 per cent of the true natural frequencies.

In the present example calculated natural frequencies are higher than the true natural frequencies. This however is not always the case and since the admissible velocities and impulses describe a system which is partly in equilibrium and partly compatible, the natural frequencies calculated can fall on either side of the true natural frequencies (see also Ref. [7]).

TABLE 3

$$
[\phi]_{12} = [\phi]^t_{21} = \frac{1}{D(1-\nu^2)}
\begin{bmatrix}
\frac{1}{3} & \frac{-\nu}{3} & 0 & \frac{1}{4} & \frac{-\nu}{4} & 0 & \frac{1}{6} & \frac{-\nu}{6} & 0 & \frac{-\nu}{5} & 0 & \frac{1}{9} & 0 & \frac{1}{8} & \frac{-\nu}{8} \\
\frac{-\nu}{3} & \frac{1}{3} & 0 & \frac{-\nu}{6} & \frac{1}{6} & 0 & \frac{-\nu}{4} & \frac{1}{4} & 0 & \frac{1}{9} & 0 & \frac{-\nu}{5} & 0 & \frac{-\nu}{8} & \frac{1}{8} \\
0 & 0 & \frac{\bar{\nu}}{4} & 0 & 0 & \frac{\bar{\nu}}{6} & 0 & 0 & \frac{\bar{\nu}}{6} & 0 & \frac{\bar{\nu}}{8} & 0 & \frac{\bar{\nu}}{8} & 0 & 0 \\
\frac{1}{4} & \frac{-\nu}{4} & 0 & \frac{1}{5} & \frac{-\nu}{5} & 0 & \frac{1}{8} & \frac{-\nu}{8} & 0 & \frac{-\nu}{6} & 0 & \frac{1}{12} & 0 & \frac{1}{10} & \frac{-\nu}{10} \\
\frac{-\nu}{4} & \frac{1}{4} & 0 & \frac{-\nu}{8} & \frac{1}{8} & 0 & \frac{-\nu}{5} & \frac{1}{5} & 0 & \frac{1}{12} & 0 & \frac{-\nu}{6} & 0 & \frac{-\nu}{10} & \frac{1}{10} \\
\frac{1}{6} & \frac{-\nu}{6} & 0 & \frac{1}{8} & \frac{-\nu}{8} & 0 & \frac{1}{9} & \frac{-\nu}{9} & 0 & \frac{-\nu}{10} & 0 & \frac{1}{12} & 0 & \frac{1}{12} & \frac{-\nu}{12} \\
0 & 0 & \frac{\bar{\nu}}{6} & 0 & 0 & \frac{\bar{\nu}}{8} & 0 & 0 & \frac{\bar{\nu}}{9} & 0 & \frac{\bar{\nu}}{10} & 0 & \frac{\bar{\nu}}{12} & 0 & 0 \\
\frac{-\nu}{6} & \frac{1}{6} & 0 & \frac{-\nu}{9} & \frac{1}{9} & 0 & \frac{-\nu}{8} & \frac{1}{8} & 0 & \frac{1}{12} & 0 & \frac{-\nu}{10} & 0 & \frac{-\nu}{12} & \frac{1}{12} \\
0 & 0 & \frac{\bar{\nu}}{6} & 0 & 0 & \frac{\bar{\nu}}{9} & 0 & 0 & \frac{\bar{\nu}}{8} & 0 & \frac{\bar{\nu}}{12} & 0 & \frac{\bar{\nu}}{10} & 0 & 0
\end{bmatrix}
$$

$(\bar{\nu} = 1+\nu)$

TABLE 4

$$
[\phi]_{22} = \frac{1}{D(1-\nu^2)}
\begin{bmatrix}
\frac{1}{5} & & & & & & & & \\
\frac{-\nu}{9} & \frac{1}{5} & & & & & & & \\
0 & 0 & \frac{\bar{\nu}}{9} & & & & & & \\
\frac{1}{6} & \frac{-\nu}{12} & 0 & \frac{1}{7} & & & & & \\
\frac{-\nu}{12} & \frac{1}{6} & 0 & \frac{-\nu}{16} & \frac{1}{7} & & & & \\
\frac{1}{10} & \frac{-\nu}{12} & 0 & \frac{1}{12} & \frac{-\nu}{15} & \frac{1}{15} & & & \\
0 & 0 & \frac{\bar{\nu}}{12} & 0 & 0 & 0 & \frac{\bar{\nu}}{15} & & \\
\frac{-\nu}{12} & \frac{1}{10} & 0 & \frac{-\nu}{15} & \frac{1}{12} & \frac{-\nu}{16} & 0 & \frac{1}{15} & \\
0 & 0 & \frac{\bar{\nu}}{12} & 0 & 0 & 0 & \frac{\bar{\nu}}{16} & 0 & \frac{\bar{\nu}}{15}
\end{bmatrix}
$$

Symmetric $(\bar{\nu} = 1+\nu)$

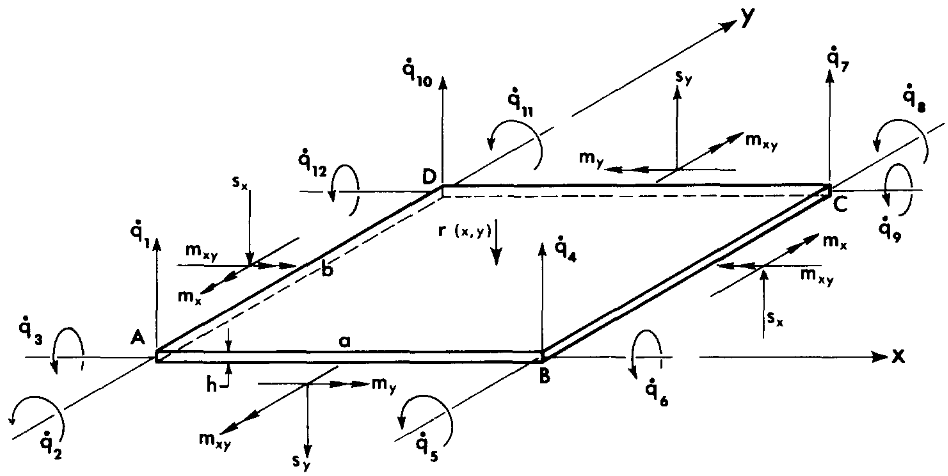

FIG. 1.

6. COMMENTS ON COMPUTING PROCEDURES

The formulation described in this paper would appear to be subject to two disadvantages. First since the frequency analysis leads to a determinantal equation and not an eigenvalue equation (as is the case in the Hamiltonian formulation when mass and stiffness matrices are obtained separately and an iteration procedure is often used to obtain the frequencies and modes), the natural frequencies and modal shapes are not found simultaneously. Second the matrix $[D]$ has to be inverted at each frequency ω in order that the dynamic mass matrix of an element can be found. Both these apparent disadvantages can be removed. The modal vectors can be obtained, relatively easily, by first partitioning the mass matrix $[\mathbf{M}]$ of the system along its first row and first column, i.e. we write $[\mathbf{M}]\{\dot{\mathbf{q}}\} = \{0\}$ as

$$\begin{bmatrix} M_{AA} & M_{AB} \\ M_{AB} & M_{BB} \end{bmatrix} \begin{bmatrix} \dot{q}_A \\ \dot{q}_B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{47}$$

where M_{AA} and $\dot{q}_A$ are scalars.

From above equation we obtain

$$M_{AA}\dot{q}_A + M_{AB}\dot{q}_B = 0$$

$$M^t_{AB}\dot{q}_A + M_{BB}\dot{q}_B = 0.$$

From the latter we have

$$\dot{q}_B = -M_{BB}^{-1} M^t_{AB}\dot{q}_A \tag{48}$$

and on substitution into the former we find

$$(M_{AA} - M_{AB}M_{BB}^{-1}M^t_{AB})\dot{q}_A = 0 \tag{49}$$

or

$$m\dot{q}_A = 0$$

where m is a scalar and it represents the dynamic mass of the entire system "viewed" from coordinate $\dot{q}_A$ only. The natural frequencies of the system are those frequencies which cause m to vanish. At these frequencies $\dot{q}_A$ can be put equal to unity and then the remainder

TABLE 5

$$
\begin{bmatrix} w_{AB} \\ w_{,xAB} \\ w_{,yAB} \\ w_{BC} \\ w_{,xBC} \\ w_{,yBC} \\ w_{CD} \\ w_{,xCD} \\ w_{,yCD} \\ w_{DA} \\ w_{,xDA} \\ w_{,yDA} \end{bmatrix}
=
\left[\begin{array}{ccc:ccc:ccc:ccc}
1-3\bar{x}^2+2\bar{x}^3 & a(\bar{x}-2\bar{x}^2+\bar{x}^3) & 0 & 3\bar{x}^2-2\bar{x}^3 & -a(\bar{x}^2-\bar{x}^3) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{6}{a}(-\bar{x}+\bar{x}^2) & 1-4\bar{x}+3\bar{x}^2 & 0 & \frac{6}{a}(\bar{x}-\bar{x}^2) & -2\bar{x}+3\bar{x}^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1-\bar{x} & 0 & 0 & \bar{x} & 0 & 0 & 0 & 0 & 0 & 0 \\
\hdashline
0 & 0 & 0 & 1-3\bar{y}^2+2\bar{y}^3 & 0 & b(\bar{y}-2\bar{y}^2+\bar{y}^3) & 3\bar{y}^2-2\bar{y}^3 & 0 & -b(\bar{y}^2-\bar{y}^3) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1-\bar{y} & 0 & 0 & \bar{y} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{6}{b}(-\bar{y}+\bar{y}^2) & 0 & 1-4\bar{y}+3\bar{y}^2 & \frac{6}{b}(\bar{y}-\bar{y}^2) & 0 & -2\bar{y}+3\bar{y}^2 & 0 & 0 & 0 \\
\hdashline
0 & 0 & 0 & 0 & 0 & 0 & 3\bar{x}^2-2\bar{x}^3 & -a(\bar{x}^2-\bar{x}^3) & 0 & 1-3\bar{x}^2+2\bar{x}^3 & a(\bar{x}-2\bar{x}^2+\bar{x}^3) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{6}{a}(\bar{x}-\bar{x}^2) & -2\bar{x}+3\bar{x}^2 & 0 & \frac{6}{a}(-\bar{x}+\bar{x}^2) & 1-4\bar{x}+3\bar{x}^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{x} & 0 & 0 & 1-\bar{x} \\
\hdashline
1-3\bar{y}^2+2\bar{y}^3 & 0 & b(\bar{y}-2\bar{y}^2+\bar{y}^3) & 0 & 0 & 0 & 0 & 0 & 0 & 3\bar{y}^2-2\bar{y}^3 & 0 & -b(\bar{y}^2-\bar{y}^3) \\
0 & 1-\bar{y} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{y} & 0 \\
\frac{6}{b}(-\bar{y}+\bar{y}^2) & 0 & 1-4\bar{y}+3\bar{y}^2 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{6}{b}(\bar{y}-\bar{y}^2) & 0 & -2\bar{y}+3\bar{y}^2
\end{array}\right]
\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \\ \dot{q}_7 \\ \dot{q}_8 \\ \dot{q}_9 \\ \dot{q}_{10} \\ \dot{q}_{11} \\ \dot{q}_{12} \end{bmatrix}
$$

TABLE 6

$$
[R]_1 = \begin{bmatrix}
0 & \frac{b}{2} & 0 & 0 & \frac{-b}{2} & 0 & 0 & \frac{-b}{2} & 0 & 0 & \frac{b}{2} & 0 \\
0 & 0 & \frac{a}{2} & 0 & 0 & \frac{a}{2} & 0 & 0 & \frac{-a}{2} & 0 & 0 & \frac{-a}{2} \\
-2 & 0 & 0 & 2 & 0 & 0 & -2 & 0 & 0 & 2 & 0 & 0 \\
\frac{-\alpha}{2} & 0 & \frac{-\alpha b}{12} & \frac{\alpha}{2} & \frac{-b}{2} & \frac{\alpha b}{12} & \frac{\alpha}{2} & \frac{-b}{2} & \frac{-\alpha b}{12} & \frac{-\alpha}{2} & 0 & \frac{\alpha b}{12} \\
0 & 0 & \frac{a}{6} & 0 & 0 & \frac{a}{3} & 0 & 0 & \frac{-a}{3} & 0 & 0 & \frac{-a}{6} \\
-1 & \frac{-a}{6} & 0 & 1 & \frac{a}{6} & 0 & -1 & \frac{-a}{6} & 0 & 1 & \frac{a}{6} & 0 \\
0 & \frac{b}{6} & 0 & 0 & \frac{-b}{6} & 0 & 0 & \frac{-b}{3} & 0 & 0 & \frac{b}{3} & 0 \\
\frac{-1}{2\alpha} & \frac{-a}{12\alpha} & 0 & \frac{-1}{2\alpha} & \frac{a}{12\alpha} & 0 & \frac{1}{2\alpha} & \frac{-a}{12\alpha} & \frac{-a}{2} & \frac{1}{2\alpha} & \frac{a}{12\alpha} & \frac{-a}{2} \\
-1 & 0 & \frac{-b}{6} & 1 & 0 & \frac{b}{6} & -1 & 0 & \frac{-b}{6} & 1 & 0 & \frac{b}{6} \\
0 & 0 & \frac{a}{12} & 0 & 0 & \frac{a}{4} & 0 & 0 & \frac{-a}{4} & 0 & 0 & \frac{-a}{12} \\
\frac{-3}{5} & \frac{-2a}{15} & 0 & \frac{3}{5} & \frac{a}{5} & 0 & \frac{-3}{5} & \frac{-a}{5} & 0 & \frac{3}{5} & \frac{2a}{15} & 0 \\
0 & \frac{b}{12} & 0 & 0 & \frac{-b}{12} & 0 & 0 & \frac{-b}{4} & 0 & 0 & \frac{b}{4} & 0 \\
\frac{-3}{5} & 0 & \frac{-2b}{15} & \frac{3}{5} & 0 & \frac{2b}{15} & \frac{-3}{5} & 0 & \frac{-b}{5} & \frac{3}{5} & 0 & \frac{b}{5} \\
\frac{-3\alpha}{20} & 0 & \frac{-b\alpha}{30} & \frac{3\alpha}{20} & \frac{-b}{6} & \frac{b\alpha}{30} & \frac{7\alpha}{20} & \frac{-b}{3} & \frac{-b\alpha}{20} & \frac{-7\alpha}{20} & 0 & \frac{b\alpha}{20} \\
\frac{-3}{20\alpha} & \frac{-a}{30\alpha} & 0 & \frac{-7}{20\alpha} & \frac{a}{20\alpha} & 0 & \frac{7}{20\alpha} & \frac{-a}{20\alpha} & \frac{-a}{3} & \frac{3}{20\alpha} & \frac{a}{30\alpha} & \frac{-a}{6}
\end{bmatrix}
$$

$$\alpha = \frac{b}{a}$$

of the modal (eigen) vector can be obtained from equation (48). The inverse of matrix $[D]$ of an element, at any frequency, can be easily obtained if $[D]$ is first expressed in its canonical form. This requires an eigenvalue analysis of $[D]$ from which a set of orthonormal eigenvectors can be obtained. If $[D]$ is of order $n \times n$ then these n vectors can be "stacked" side by side to form a modal matrix $[\theta]$. It can then be shown that [13]

$$[D] = [\theta][\Lambda - \omega^2 I][\theta]^t \tag{50}$$

where $[I]$ is an $n \times n$ unit matrix and

$$\Lambda = \text{Dia. } [\omega_1^2 \omega_2^2 \ldots \omega_n^2]$$

TABLE 7

$$[R]_2 = \begin{bmatrix} 0 & 0 & 0 & \alpha & \frac{-b}{2} & \frac{\alpha b}{6} & \alpha & \frac{-b}{2} & \frac{-b\alpha}{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\alpha} & \frac{-a}{6\alpha} & \frac{-a}{2} & \frac{1}{\alpha} & \frac{a}{6\alpha} & \frac{-a}{2} \\ 0 & 0 & 0 & 1 & 0 & \frac{b}{6} & \frac{-1}{2} & \frac{-a}{6} & \frac{-b}{6} & 1 & \frac{a}{6} & 0 \\ 0 & 0 & 0 & \frac{3\alpha}{2} & \frac{-b}{2} & \frac{\alpha b}{4} & \frac{3\alpha}{2} & \frac{-b}{2} & \frac{-\alpha b}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{2\alpha} & \frac{-a}{4\alpha} & \frac{-a}{2} & \frac{3}{2\alpha} & \frac{a}{4\alpha} & \frac{-a}{2} \\ 0 & 0 & 0 & \frac{3\alpha}{10} & \frac{-b}{6} & \frac{b\alpha}{15} & \frac{7\alpha}{10} & \frac{-b}{3} & \frac{-b\alpha}{10} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \frac{b}{6} & \frac{2}{5} & \frac{-a}{5} & \frac{-b}{6} & \frac{3}{5} & \frac{2a}{15} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{7}{10\alpha} & \frac{-a}{10\alpha} & \frac{-a}{3} & \frac{3}{10\alpha} & \frac{a}{15\alpha} & \frac{-a}{6} \\ 0 & 0 & 0 & \frac{3}{5} & 0 & \frac{2b}{15} & \frac{2}{5} & \frac{-a}{6} & \frac{-b}{5} & 1 & \frac{a}{6} & 0 \end{bmatrix}$$

$$\alpha = \frac{b}{a}$$

ω_i^2 being the ith eigenvalue of $[D]$. From equation (50) it can be seen that to obtain $[D]^{-1}$ one need only obtain the scalar inverse of the diagonal elements of $[\Lambda - \omega^2 I]$. Thus the dynamic mass matrix of the jth element can now be written as

$$\begin{aligned}[M]_j &= [R]_j^t[\theta]_j[\Lambda - \omega^2 I]_j^{-1}[\theta]_j^t[R]_j \\ &= [H]_j^t[E]_j[H]_j\end{aligned}$$

TABLE 8

No. of Elements	Degrees of Freedom	μ_1	μ_2	μ_3	μ_4	μ_5
4	3	680·2	1337	5508	–	–
8	9	630·3	1116·2	2527	5188	–
12	18	620·1	1087·5	2250	4290	–
24	45	613·2	1053·5	2136	4200	5330
Rayleigh–Ritz		608	1020	2070	4110	5100

$$\mu = \frac{\rho h \omega^2 l_x^4}{D}.$$

where

$$[H]_j = [\theta]_j^t[R]_j$$

and

$$[E]_j = [\Lambda - \omega^2 I]_j^{-1},$$

a diagonal matrix. Now the mass matrix of the entire system can be obtained by transforming to the global coordinates by means of connection matrix $[J]$ i.e.

$$[\mathbf{M}] = [J]^t \begin{bmatrix} H_1^t & & & \\ & H_2^t & & \\ & & \ddots & \\ & & & H_p^t \end{bmatrix} \begin{bmatrix} E_1 & & & \\ & E_2 & & \\ & & \ddots & \\ & & & E_p \end{bmatrix} \begin{bmatrix} H_1 & & & \\ & H_2 & & \\ & & \ddots & \\ & & & H_p \end{bmatrix} [J]\cdot$$

or

$$[\mathbf{M}] = [\Gamma]^t[E][\Gamma] \tag{51}$$

where

$$[\Gamma] = \text{Dia.}[H_1 H_2 \ldots H_p]$$
$$[E] = \text{Dia.}[E_1 E_2 \ldots E_p].$$

From equation (51) it is apparent that the dynamic mass matrix of the system can be obtained, for any frequency ω, by a simple matrix multiplication routine. By appropriate partitioning of $[\Gamma]$ it is also possible to express m [see equation (49)] explicitly in terms of ω^2. In Ref. [14] this has been done and further the slope of m v. ω^2 has also been obtained explicitly in terms of the previously defined matrices. Having expressed m and its slope in terms of ω^2 one can then use the Newton iterative procedure to obtain the zeros of m vs. ω^2 plot efficiently. Examples and details of this procedure are given in Ref. [14].

7. CONCLUDING COMMENTS

We have extended the hybrid finite element formulation of Pian to dynamic problems and we have illustrated the procedure by calculating the first few natural frequencies of a plate in flexural motion.

The formulation given here is based on a variational principle in which admissible impulses satisfy equations of equilibrium inside the element while admissible boundary velocities guarantee interelement compatibility. Admissible impulses and velocities however will not in general satisfy the compatibility equations inside the elements and also there will generally be a discontinuity of velocities from one element to the next. Nor will equilibrium conditions be completely satisfied at the interelement boundaries by the admissible velocities and impulses. The extremisation process will tend to satisfy those conditions which are not explicitly satisfied by admissible impulses and velocities, implicitly.

The general solution presents a mixture of partly compatible and partly equilibrating systems and hence the natural frequencies calculated will not be bound below or above by the true natural frequencies i.e. no assurance can be given as to whether the calculated frequencies are lower or higher than the true natural frequencies.

In the present method the kinematic boundary conditions are satisfied explicitly at the outset by means of assumed boundary velocities. It is interesting to note that while the dynamic boundary conditions will tend to be satisfied via the extremization process, one can sometimes satisfy the dynamic boundary conditions explicitly by the proper choice of assumed impulse modes [2].

Finally it is of interest to note that a dual hybrid method, for static analysis, was presented by Jones [12] in which displacement functions are used within the element and stress functions along the boundaries of the element. Extension of Jone's method to dynamic problems is presently being examined.

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Абстракт—С помощью варияционного принципа, расщряемся способ смешанного конечного элемента, предложенный Планом для расчета сплошной среды.

Формулировка иллюстрируется, затем, расчетом частот пластинки, находящейся в изгибном движению.